Proof of the impossibility of probabilistic induction

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In what follows I restate and simplify the proof of the impossibility of probabilistic induction given in [1, 2]. Other proofs are possible (cf. [1]).

1 Logical Entailment

Given two statements x, y we write $x \vdash y$ if x *entails* y. For example, if

$$
\begin{aligned}
x &= \text{``All men are mortal''} \\
y &= \text{``Socrates is a man''} \\
z &= \text{``Socrates is mortal''}
\end{aligned}
$$

we can write $(x \wedge y) \vdash z$. Entailment is a "truth broadcasting" operation that allows the truth of the premises to flow to the conclusion.

We also have the following simple theorem.

Theorem 1 *If $x \vdash y$ then $p(y|x) = 1$.*

For example if

$$
\begin{aligned}
x &= \text{``All swans are white''} \\
y &= \text{``All swans in Austria are white''}
\end{aligned}
$$

Then $x \vdash y$ and $p(y|x) = 1$ reads "The probability that all swans in Austria are white given that all swans are white is 1."

2 The Problem Of Induction

We describe the classical problem of induction by first describing (the non-problem of) deduction. We have a "general law" statement G which is assumed to hold across time and space, and we have a set of observed evidence statements $e_1, e_2, ..., e_N$ that are assumed to be true in a particular spatio-temporal region. We refer to the conjunction of the observed evidence as $E = (e_1 \wedge e_2 \wedge ...)$. By definition the general law G logically entails each of the evidence statements

$$G \vdash e_i \qquad\qquad \forall e_i \in E \qquad\qquad (1)$$

For example, consider evidence statements

$$e_1 = \textit{"All observed swans in Austria in 1796 are white"}$$
$$e_2 = \textit{"All observed swans in Austria in 1797 are white"}$$
$$...$$

And consider the generalization

$$G = \textit{"All swans are white"}$$

Then $G \vdash E$ is logically valid and referred to as "deductive inference". However, $E \vdash G$ is logically *invalid*. This is the classical "problem of induction". It states that no amount of evidence statements e_i can *logically* justify a generalization G.

3 Probabilistic Solution to the Problem of Induction

The probabilistic solution states that while the truth of G cannot be logically established, the *probability* of any general law G can be established and increases with the accumulation of favorable evidence. Therefore, while we cannot say G is true, we can say G is *probably* true. The argument is as follows.

Let G and E be as above. Let $0 < P(G) < 1$ be our prior belief in G and let $0 < P(E) < 1$ be the probability of the evidence. Because $G \vdash E$ we have $P(E|G) = 1$. Therefore using Bayes rule we arrive at the following:

$$P(G|E) = \frac{P(E|G)P(G)}{P(E)} = \frac{P(G)}{P(E)}$$

And because $0 < \mathrm{P(E)} < 1$

$$P(G|E) > P(G) \qquad\qquad (2)$$

Therefore the probability of G increases with favorable evidence E. This seemingly justifies a belief in probabilistic induction. In the next section it will be shown that despite this seeming plausibility, probabilistic induction is impossible.

4 Proof of the Impossibility of the Probabilistic Solution to the Problem of Induction

Again let $E = (e_1 \wedge e_2 \wedge ...)$ where

$$e_1 = \text{``All observed swans in Austria in 1796 are white''}$$
$$e_2 = \text{``All observed swans in Austria in 1797 are white''}$$
$$...$$

Now instead consider two competing generalizations of the evidence E

G_g = *"All swans are white"*
G_b = *"All swans are violet except in Austria where they are white"*.

G_g is a typical "good" generalization while G_b is a "bad", or "anti-inductive" generalization of the evidence. Being Bayesian, we can give the prior probabilities of $P(G_g)$ and $P(G_b)$ any values between 0 and 1 we like.

We will consider the two ratios

$$R_{\text{prior}} = \frac{P(G_g)}{P(G_b)} \qquad\qquad R_{\text{posterior}} = \frac{P(G_g|E)}{P(G_b|E)}$$

Because any generalization by definition entails the evidence, we have $P(E|G_g) = P(E|G_b) = 1$. Therefore we have the following theorem

3

Theorem 2

$$R_{posterior} = \frac{P(E|G_g)P(G_g)/P(E)}{P(E|G_b)P(G_b)/P(E)} = R_{prior} \tag{3}$$

This shows that inductive learning can never favor one generalization over another. Despite raising the probability according to (2), it raises the probability of *all* generalizations, even anti-inductive ones such as G_b. Or, in Popper's words [2]:

"Theorem (2) is shattering. It shows that the favourable evidence E, even though it raises the probability according to (2), nevertheless, against first impressions, leaves everything precisely as it was. It can never favour (G_g) rather than (G_b). On the contrary, the order which we attached to our hypotheses before the evidence remains. It is unshakable by any favourable evidence. The evidence cannot influence it."

Therefore probabilistic induction cannot favor inductive generalizations over anti-inductive generalizations, and we conclude that probabilistic induction is impossible.

5 Commentary

The proof relies on the fact that, for any generalization G_i of the evidence E, the likelihood $P(E|G_i) = 1$. This is the formal condition for induction – to induce general laws of nature from observations. If one wished to refute the proof, they would have to either claim:

1. G_i isn't a general law, which means probabilistic induction is not capable of inducing general laws, or

2. G_i is a general law but that $P(E|G_i) \neq 1$. Stated using our example, this says: "the probability that all swans in Austria are white is not 1, despite the fact that all swans are white." In other words, the general law G_i is not a general law, which is a contradiction.

This shows that probability calculus is not capable of discovering (i.e. inducing) general laws of nature from data. Given that human beings *are* capable of discovering general laws of nature from data, this further shows that the products of human cognition are not products of the probability calculus.

References

[1] Karl Popper. The non existence of probabilistic inductive support. *1985*.

[2] Karl Popper. Realism and the aim of science: From the postscript to the logic of scientific discovery. *1992*.